Indefinite Integration

-Integration works around the idea of working the derivative "backwards" called **antidifferentiation.**

-A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

$F(x) = x^3$ becomes $\frac{d}{dx}\left[x^3\right] = 3x^2$

-A differential equation in x and y is an equation that involves x and y and the derivatives of y

-**Ex**

$y' = 3x$ $y' = x^2 + 1$

Example

-Find the general solution to the differential equation $y' = 2$.

$$y = 2x + C$$

Notation

-When solving the form $\frac{dy}{dx} = f(x)$ it is sometimes convenient to write in the form

$$dy = f(x)dx$$

-The operation of finding all solutions of this equation is called **antidifferentiation** or **indefinite integration** and is denoted $\int$

variable of integration

$$y = \int f\left(x\right)\,dx = F\left(x\right) + C$$

constant of integration

integrand

Basic Integration Rules

$\int F'\left(x\right)dx = F\left(x\right) + C$ “integration is the “inverse” of differentiation”

$\frac{d}{dx}\left[\int f\left(x\right)dx\right] = f\left(x\right)$ “differentiation is the “inverse” of integration”

Example

Describe the antiderivative of $3x$.

$\int 3x\,dx = 3\int x\,dx$ ‘constant multiple

$= 3\int x^1\,dx$

$= 3\left(\frac{x^2}{2}\right) + C$ ‘Power Rule

$= \frac{3}{2}x^2 + C$ ‘Simplify

Steps

-Original Integral $\Rightarrow$ Rewrite $\Rightarrow$ Integrate $\Rightarrow$ Simplify

Examples

$$\int \frac{1}{x^3}\, dx \Rightarrow x^{-3}dx \Rightarrow \frac{x^{-2}}{-2} + C \Rightarrow -\frac{1}{2x^2} + C$$

$$\int \sqrt{x}\, dx \Rightarrow \int x^{1/2}dx \Rightarrow \frac{x^{3/2}}{3/2} + C \Rightarrow \frac{2}{3}x^{3/2} + C$$

$$\int 2\sin(x)dx \Rightarrow 2\int \sin(x)dx \Rightarrow 2(-\cos(x)) + C \Rightarrow -2\cos(x) + C$$

Example-Integrating Polynomials

$$\int dx = \int 1\, dx$$

$$= x + C$$

$$\int (x+2)\, dx = \int x\, dx + \int 2\, dx$$

$$= \frac{x^2}{2} + C_1 + 2x + C_2$$

$$= \frac{x^2}{2} + 2x + C$$

$$\int (3x^4 - 5x^2 + x)\, dx$$

$$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$$

Example

$$\int \frac{x+1}{\sqrt{x}}\, dx$$

$$= \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right) dx$$

$$= \int \left(x^{1/2} + x^{-1/2} \right) dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3} x^{3/2} + 2x^{1/2} + C$$

NOTE: Just like differentiation you can't just integrate numerator and denominator.

Example

$$\int \frac{\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{1}{\cos x} \right)\left(\frac{\sin x}{\cos x} \right) dx$$

$$= \int \sec x \tan x \, dx$$

$$= \sec x + C$$

Initial Conditions and Particular Solutions

$$y = \int \left(3x^2 - 1\right) dx = x^3 - x + C$$ 'General Solution

If you have a value $y = F\left(x\right)$ for a value of x called the **initial condition** you can find the **particular solution.**

i.e. consider the point $\left(2,4\right)$

$$F\left(x\right) = x^3 - x + C$$

$$F\left(2\right) = 4$$

$$F\left(2\right) = 8 - 2 + C = 4$$

$$C = -2$$

So,

$F\left(x\right) = x^3 - x - 2$ is a particular solution.

Example

Find the general solution of $F'\left(x\right) = \frac{1}{x^2}, x > 0$ and the particular solution that satisfies $F\left(1\right) = 0$.

$$F\left(x\right) = \int \frac{1}{x^2} dx$$

$$= \int x^{-2} dx$$

$$= \frac{x^{-1}}{-1} + C$$

$$\frac{-1}{x} + C, x > 0$$

$$F(1) = \frac{-1}{1} + C = 0$$

$$C = 1$$

$$F(x) = \frac{-1}{x} + 1, x > 0$$

Example

A ball is thrown upward with an initial velocity of 64 ft/sec from an initial height of 80 ft.

Find the position function giving the height s as a function if time t.

When does the ball hit the ground?

Let $t = 0$ be initial time.

$s(0) = 80$ 'Initial Height

$s'(0) = 64$ 'Initial velocity

Using -32 ft/sec^2 as gravity:

$$s''(t) = -32$$

$$s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1$$

Use initial velocity:

$$s'(0) = 64$$

$$s'(t) = -32t + C_1$$

$$-32(0) + C_1 = 64$$

$$C_1 = 64$$

$$s'(t) = -32t + 64$$

Next by integrating $s'(t)$

$$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt$$

$$= -16t^2 + 64t + C_2$$

Using initial height:

$$s(0) = 80$$

$$s(t) = -16t^2 + 64t + C_2$$

$$-16(0)^2 + 64(0) + C_2 = 80$$

$$C_2 = 80$$

$$s(t) = -16t^2 + 64t + 80$$

Solve the position function for where the ball hit the ground.

$$s(t) == -16t^2 + 64t + 80 = 0$$

$$-16(t+1)(t-5) = 0$$

$$t = -1, 5$$

$$t = 5 \text{ sec}$$